THE HAUSDORFF-BESICOVICH DIMENSION OF THE LEVEL SETS OF PERRON'S MODULAR FUNCTION

BY

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1. **Introduction.** The modular function M was introduced by O. Perron in [5]. $M(\xi)$ (for irrational ξ) is defined by the property that, for any $\varepsilon > 0$, the inequality

$$\left|\xi - \frac{p}{q}\right| < \frac{1+\varepsilon}{M(\xi)q^2}$$

has infinitely many integer solutions (p, q) while the inequality

$$\left|\xi - \frac{p}{q}\right| < \frac{1-\varepsilon}{M(\xi)q^2}$$

has only finitely many. Let

$$\xi = \left[x_1, x_2, \cdots\right] = \frac{1}{x_1+}\ \frac{1}{x_2+}\ \cdots$$

be the continued fraction expansion of ξ (we will only use the continued fraction expansion for numbers in the open interval $(0,1)$) and set

$$M_k(\xi) = x_k + \left(\frac{1}{x_{k-1}+}\ \frac{1}{x_{k-2}+}\ \cdots + \frac{1}{x_1}\right) + \left(\frac{1}{x_{k-1}+}\ \frac{1}{x_{k+2}+}\ \cdots\right).$$

It is easily shown (see [5]) that

$$M(\xi) = \limsup_{k\to\infty} M_k(\xi).$$

For every positive number γ let

$$L(\gamma) = [\xi \mid M(\xi) = \gamma]$$

be the level set of M at γ. The theorem of this paper provides an estimate of the Hausdorff-Besicovich dimension of $L(\gamma)$ for sufficiently large γ. The Hausdorff-Besicovich dimension of a set S, which we will write $\dim(S)$, is defined as follows: let (I_i) be a covering of S by intervals, and let $|I_i|$ be the length of I_i; then $\delta = \text{l.u.b.}\, |I_i|$ is called the norm of the covering;

$$\Gamma(\lambda, S) = \lim_{\delta\to\infty} \text{g.l.b.} \sum |I_i|^\lambda,$$

where the greatest lower bound is taken over all coverings of norm δ, is the

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λ-dimensional Hausdorff measure of S and $\dim(S)$ is the number such that, for every positive ε,

$$\Gamma(\dim(S) - \varepsilon, S) = \infty$$

and

$$\Gamma(\dim(S) + \varepsilon, S) = 0.$$

We will write E_N for the set of those numbers in (0, 1) whose continued fraction expansions involve no integer bigger than N, s_N for the smallest number in E_N and l_N for the largest. It is easily seen that

$$s_N = [N, 1, N, 1, \cdots] = \frac{(N^2+4N)^{1/2} - N}{2N}$$

and

$$l_N = [1, N, 1, N, \cdots] = \frac{(N^2+4N)^{1/2} - N}{2}.$$

The following facts on continued fraction expansions can be found in [3]. Let $\xi = [x_1, x_2, \cdots]$. Then the integers $P_k(\xi)$ and $Q_k(\xi)$ defined by

$$P_0(\xi) = 0, P_1(\xi) = 1,$$

$$Q_0(\xi) = 1, Q_1(\xi) = x_1,$$

$$P_{k+1}(\xi) = x_{k+1} P_k(\xi) + P_{k-1}(\xi),$$

$$Q_{k+1}(\xi) = x_{k+1} Q_k(\xi) + Q_{k-1}(\xi)$$

satisfy

$$\frac{P_k(\xi)}{Q_k(\xi)} = \frac{1}{x_1+} \frac{1}{x_2+} \cdots + \frac{1}{x_k},$$

$$P_{k-1}(\xi) Q_k(\xi) - P_k(\xi) Q_{k-1}(\xi) = (-1)^k.$$

If we set $\eta = [x_{k+1}, x_{k+2}, \cdots]$ then

$$\xi = \frac{(x_k + \eta) P_{k-1}(\xi) + P_{k-2}(\xi)}{(x_k + \eta) Q_{k-1}(\xi) + Q_{k-2}(\xi)}.$$

If $\xi' = [x'_1, x'_2, \cdots]$ where $x_i = x'_i$ for $i < k$ and if we set $\eta' = [x'_{k+1}, x'_{k+2}, \cdots]$ then

$$(1) \quad \begin{aligned} |\xi - \xi'| &= \frac{|x_k + \eta - x' - \eta'|}{((x_k + \eta) Q_{k-1}(\xi) + Q_{k-2}(\xi))((x'_k + \eta') Q_{k-1}(\xi) + Q_{k-2}(\xi))} \\ &= \frac{|x_k + \eta - x'_k - \eta'|}{(Q_k(\xi) + \eta Q_{k-1}(\xi))(Q_k(\xi') + \eta' Q_{k-1}(\xi'))}. \end{aligned}$$

$Q_k(\xi)$ is an increasing function of k and satisfies

$$\left(\prod_{k=j+1}^{j+l} x_k\right) Q_j(\xi) \leqq Q_{j+l}(\xi) \left(\prod_{k=j+1}^{j+l} (x_k+1)\right) Q_j(\xi). \tag{2}$$

2. **Preliminary lemmas.** We define intervals A_N, B_N, and C_N by;

$$\begin{aligned} A_4 &= \left[5 + l_4 - \frac{1}{6+l_4},\ 5 + 2l_4\right), \\ B_4 &= [5 + 2l_4,\ 5 + 2l_5), \\ A_N &= [N + 2l_N,\ N + 1 + 2l_4), \qquad N \geqq 5, \\ B_N &= [N + 1 + 2l_4,\ N + 1 + 2l_{N+1}), \qquad N \geqq 5, \\ C_N &= A_N \cup B_N, \qquad N \geqq 4. \end{aligned}$$

Our first lemma is a direct corollary of a theorem of Marshall Hall.

LEMMA 1. *If γ is in C_N, $N \geqq 4$ we can write*

$$\gamma = F(\gamma) + \alpha + \beta$$

where α and β are in E_4 and $F(\gamma)$ is $N + 1$ for γ in A_N and is $N + 2$ for γ in B_N.

Proof. By Theorem 3.1 of [1] every number γ in $[M + 2s_4,\ M + 2l_4]$ can be written in the form $M + \alpha + \beta$ with α and β in E_4. Straightforward computation shows that $[N + 1 + 2s_4,\ N + 1 + 2l_4]$ contains A_N and that $[N + 2 + 2s_4,\ N + 2 + 2l_4]$ contains B_N for all $N \geqq 4$.

Suppose now that $\gamma = F(\gamma) + \alpha + \beta$ is in C_N with $\alpha = [a_1, a_2, \cdots]$ and $\beta = [b_1, b_2, \cdots]$ in E_4. We choose a sequence (p_i) of integers satisfying

$$p_1 \geqq 3$$

and

$$\lim_{k\to\infty} \frac{k^2}{S_k} = 0$$

where

$$\begin{aligned} S_k &= 0 \quad \text{if } k = 0 \\ &= \sum_{i=1}^{k} p_i \quad \text{if } k > 0. \end{aligned}$$

For any $\xi = [x_1, x_2, \cdots]$ in E_N we define $\phi(\xi) = [y_1, y_2, \cdots]$ by setting

$$\begin{aligned} y_j &= x_{j-n(n-1)} \quad \text{if} \quad S_{n-1} + n(n-1) < j \leqq S_n + n(n-1), \\ &= a_{n+1-l} \quad \text{if} \quad j = S_n + n(n-1) + l, \qquad l = 1, \cdots, n, \\ &= F(\gamma) \quad \text{if} \quad j = S_n + n^2 + 1, \\ &= b_l \quad \text{if} \quad j = S_n + n^2 + 1 + l, \qquad l = 1, \cdots, n-1, \end{aligned}$$

for $n = 1, 2, \cdots$. If we set

$$\pi(j) = j + n(n-1) \quad \text{if } S_{n-1} < j \leqq S_n$$

then $x_j = y_{\pi(j)}$, j and $\pi(j)$ have the same parity and

$$1 \leqq \frac{\pi(j)}{j} = 1 + \frac{n(n-1)}{j} \leqq 1 + \frac{n(n-1)}{S_{n-1}} \to 1.$$

LEMMA 2. *ϕ is an order preserving map of E_N into $L(\gamma)$.*

Proof. If $y_k = 4$

$$M_k(\phi(\xi)) \leqq \max\left(4 + l_4 + \frac{1}{1 + (1/F(\gamma) + l_4)}, 4 + l_4 + l_N\right) + O(1)$$

$$= 5 + l_4 - \frac{1}{F(\gamma) + 1 + l_4} + O(1),$$

and if $y_k = N$

$$M_k(\phi(\xi)) \leqq N + 2l_N + O(1).$$

If follows from this that

$$\limsup_{k\to\infty} M_k(\phi(\xi)) = \lim_{n\to\infty} M_{S_n+n^2+1}(\phi(\xi))$$

$$= F(\gamma) + \alpha + \beta = \gamma,$$

i.e. that $\phi(\xi)$ is in $L(\gamma)$.

Now suppose that $\xi = [x_1, x_2, \cdots] < \xi' = [x_1', x_2', \cdots]$ and let k be the first integer for which $x_k \neq x_k'$. Then either k is odd and $x_k' < x_k$ or k is even and $x_k' > x_k$. If $\phi(\xi) = [y_1, y_2, \cdots]$ and $\phi(\xi') = [y_1', y_2', \cdots]$ then $y_j = y_j'$ for $j < \pi(k)$ and either $\pi(k)$ is odd and $y'_{\pi(k)} = x_k' < x_k = y_{\pi(k)}$ or $\pi(k)$ is even and $y'_{\pi(k)} = x_k' > x_k = y_{\pi(k)}$. Hence $\phi(\xi) < \phi(\xi')$.

Set, for $k \geqq 2$,

$$\rho(k) = \pi(k) - k + \pi(k-1) - (k-1).$$

Note that ρ is an increasing function of k and that

$$\lim_{k\to\infty} \frac{\rho(k)}{k} = 2\lim_{k\to\infty}\left(\frac{\pi(k)}{k} - 1\right) = 0.$$

LEMMA 3. *For $k \geqq 2$*

$$Q_k(\xi) \leqq Q_{\pi(k)}(\phi(\xi)) \leqq (N+3)^{\rho(k)} Q_k(\xi).$$

Proof. Since $p_1 \geqq 3$ the lemma holds for $k = 2$ or 3. Suppose it holds for all $j \leqq k$. Then if $\xi = [x_1, x_2, \cdots]$ is in E_N and $\phi(\xi) = [y_1, y_2, \cdots]$ we have

$$\begin{aligned} Q_{\pi(k+1)}(\phi(\xi)) &= y_{\pi(k+1)}Q_{\pi(k+1)-1}(\phi(\xi)) + Q_{\pi(k+1)-2}(\phi(\xi)) \\ &\geqq x_{k+1}Q_{\pi(k)}(\phi(\xi)) + Q_{\pi(k-1)}(\phi(\xi)) \\ &\geqq x_{k+1}Q_k(\xi) + Q_{k-1}(\xi) = Q_{k+1}(\xi) \end{aligned}$$

so the first inequality holds for $k+1$.

Now suppose that $\pi(k \pm 1) = \pi(k) \pm 1$ and hence that $\rho(k+1) = \rho(k)$. Then

$$\begin{aligned} Q_{\pi(k+1)}(\phi(\xi)) &= y_{\pi(k+1)}\, Q_{\pi(k)}(\phi(\xi)) + Q_{\pi(k-1)}(\phi(\xi)) \\ &\leqq (N+3)^{\rho(k)}(x_{k+1}Q_k(\xi) + Q_{k-1}(\xi)) \\ &= (N+3)^{\rho(k)}\, Q_{k+1}(\xi). \end{aligned}$$

If $\pi(k+1) = \pi(k) + 2n + 1$, i.e. if $k = S_n$, then $\pi(k-1) = \pi(k) - 1$ and $\rho(k+1) = \rho(k) + 2n$. We have, using (2),

$$\begin{aligned} Q_{\pi(k+1)}(\phi(\xi)) &= y_{\pi(k+1)}\, Q_{\pi(k+1)-1}(\phi(\xi)) + Q_{\pi(k+1)-2}(\phi(\xi)) \\ &\leqq (N+3)^{2n}(x_{k+1}\, Q_{\pi(k)}(\phi(\xi)) + Q_{\pi(k-1)}(\phi(\xi))) \\ &\leqq (N+3)^{2n+\rho(k)}(x_{k+1}\, Q_k(\xi) + Q_{k-1}(\xi)) \\ &= (N+3)^{\rho(k+1)}\, Q_{k+1}(\xi). \end{aligned}$$

The case $k-1 = S_n$, $\pi(k+1) = \pi(k) + 1$, $\pi(k-1) = \pi(k) - 2n - 1$, $\rho(k+1) = \rho(k) + 2n$ is handled similarly.

Lemma 4. *If $\xi = [x_1, x_2, \cdots]$ and $\xi' = [x'_1, x'_2, \cdots]$ are in E_M and $x_i = x'_i$ for $i < k$ but $x_k \neq x'_k$ then*

$$|\xi - \xi'| \geqq \frac{1 - (l_M - s_M)}{4Q_k(\xi)Q_k(\xi')} \geqq \frac{1 - (l_M - s_M)}{4(M+1)^{2k}}.$$

Proof. Setting $\eta = [x_{k+1}, x_{k+2}, \cdots]$ and $\eta' = [x'_{k+1}, x'_{k+2}, \cdots]$ and using formulas (1) and (2) we have

$$\begin{aligned} |\xi - \xi'| &= \frac{|x_k + \eta - x'_k - \eta'|}{(Q_k(\xi) + \eta Q_{k-1}(\xi))\,(Q_k(\xi') + \eta' Q_{k-1}(\xi'))} \\ &\geqq \frac{1 - (l_M - s_M)}{4Q_k(\xi)\, Q_k(\xi')} \geqq \frac{1 - (l_M - s_M)}{4(M+1)^{2k}}. \end{aligned}$$

Lemma 5. *If $\xi = [x_1, x_2, \cdots]$ and $\xi' = [x'_1, x'_2, \cdots]$* are *in E_N and $x_i = x'_i$ for $i < k$ but $x_k \neq x'_k$ then there is an A depending only on N such that*

$$|\phi(\xi) - \phi(\xi')| \geqq A(N+3)^{-2\rho(k)}|\xi - \xi'|.$$

Proof. By Lemmas 3 and 4

$$|\phi(\xi)-\phi(\xi')| \geqq \frac{1-(l_{N+2}-s_{N+2})}{4Q_{\pi(k)}(\phi(\xi))Q_{\pi(k)}(\phi(\xi'))}$$

$$\geqq \frac{1-(l_{N+2}-s_{N+2})}{4(N+3)^{2\rho(k)}} \quad \frac{1}{Q_k(\xi)Q_k(\xi')}.$$

But, if $\eta = [x_k, x_{k+1}, \cdots]$ and $\eta' = [x'_k, x'_{k+1}, \cdots]$, then

$$|\xi-\xi'| = \frac{|x_k+\eta-(x'_k+\eta')|}{(Q_k(\xi)+\eta Q_{k-1}(\xi))\,(Q_k(\xi')+\eta' Q_{k-1}(\xi'))}$$

$$\leqq \frac{N}{Q_k(\xi)\,Q_k(\xi')}$$

so the lemma holds with

$$A = (1-(l_{N+2}-s_{N+2}))/4N\ .$$

3. **The dimension of** $L(\gamma)$. In §7 of [4] a probability measure ω^* on $[0,1]$ was defined for every pair of positive integers M_1 and M_2 with $M_1 > M_2+1$. We take ω_N^* to be one that corresponds to $M_1 = N+1$ and $M_2 = 1$ so that $\omega_N^*(E_N) = 1$. Let

$$h_N = -2\int_0^1 \log t\ \omega_N^*(dt).$$

It is clear from the estimates in [4] that $0 < h_N < \infty$. For every $\varepsilon > 0$ and integer l set

$$E'_N(\varepsilon, l) = \left[\ \xi \,\middle|\, \xi\in E_N \text{ and } \left|\frac{P_k(\xi)}{Q_k(\xi)}-\xi\right| \leqq \exp\left[-k(h_N-\varepsilon)\right] \text{ for all } k\geqq l\right].$$

If

$$E'_N = \lim_{\varepsilon\to 0}\ \lim_{l\to\infty} E'_N(\varepsilon, l)$$

then it follows from Theorems 2.2 and 7.1 of [4] that

$$\dim(E'_N) \geqq \frac{2}{h_N}\left[\ \log N - \int_0^1 \log((N-1)t+1)\,\frac{d\omega}{d\lambda}(t)\,dt\right] > 0$$

where

$$\frac{d\omega}{d\lambda}(t) = \frac{N-1}{\log\left(\dfrac{N(2N+1)}{(N+1)^2}\right)}\left(\frac{1}{N+1+(N-1)t} - \frac{1}{N^2+1+(N-1)t}\right).$$

The estimate in Theorem 8.1 of [4] gives

$$\dim(E'_N) \geqq 1 - \frac{2}{\tau(N-1)} + O\left(\frac{1}{N^2}\right)$$

where $\tau = \pi^2/(6 \log 2)$ is Khintchine's constant.

THEOREM. *If* $\gamma \geqq (7/2) + (7/4)\, 2^{1/2}$ *then* $\dim(L(\gamma)) > 0$. *For* γ *in* C_N

$$\dim(L(\gamma)) \geqq 1 - \frac{2}{\tau(N-1)} + O\left(\frac{1}{N^2}\right).$$

For γ *in* C_N, $N \geqq 6$

$$\dim(L(\gamma)) \leqq 1 - \frac{1}{8(N+2)\log(N+2)}.$$

Proof. Set $\gamma = F(\gamma) + \alpha + \beta$ as in the previous section. We will first show that $\dim(\phi(E'_N)) \geqq \dim(E'_N)$ which will establish the first and second assertions. For small enough ε and large enough l we can find $\lambda < 1$ such that

$$k\left[\lambda(h_N - \varepsilon) - 2\frac{\rho(k)}{k}\log(N+3)\right] + \log A - \lambda \log(2e^{h_N}) > 0$$

for all $k \geqq l$. By Lemma 4 with $M = N+2$ we can choose δ so small that $|\phi(\xi) - \phi(\eta)| \leqq \delta$ implies that $\phi(\xi)$ and $\phi(\eta)$ have the same first $\pi(l-1)$ continued fraction coefficients and hence that ξ and η have the same first $l-1$ coefficients. Let (I_i) be a covering of $\phi(E'_N(\varepsilon, l))$ of norm at most δ. We can assume that each I_i is the smallest interval containing $I_i \cap \phi(E'_N(\varepsilon, l))$. Let $J_i = \phi^{-1}(I_i)$. Since ϕ preserves order, there are sequences (ξ_j) and (η_j) contained in J_i such that

$$\frac{|I_i|}{|J_i|^{1+\lambda}} = \lim_{j\to\infty} \frac{|\phi(\xi_j) - \phi(\eta_j)|}{|\xi_j - \eta_j|^{1+\lambda}}$$

$$\geqq \limsup_{j\to\infty} \frac{A}{(N+3)^{2\rho(k_j)}} \frac{1}{|\xi_j - \eta_j|^{\lambda}}$$

where $k_j \geqq l$ is the first place where the continued fraction coefficients of ξ_j and η_j differ. But

$$|\xi_j - \eta_j| \leqq \left|\xi_j - \frac{P_{k_j-1}(\xi_j)}{Q_{k-1}(\xi_j)}\right| + \left|\frac{P_{k_j-1}(\eta_j)}{Q_{k_j-1}(\eta_j)} - \eta_j\right|$$

$$\leqq 2\exp[-(k_j-1)(h_N - \varepsilon)] \leqq 2e^{h_N}\exp[-k_j(h_N-\varepsilon)]$$

so that

$$\frac{|I_i|}{|J_i|^{1+\lambda}} \geqq \limsup_{j\to\infty} \frac{A}{(2\exp(h_N))^2} \exp\left\{ k_j \left[\lambda(h_N - \varepsilon) - \frac{2\rho(k_j)}{k_j} \log(N+3) \right] \right\}$$

$$\geqq 1.$$

Now (J_i) is a covering of $E'_N(\varepsilon, l)$ of norm at most $\delta^{1/(1+\lambda)}$ so for all small enough δ and

$$\mu < \frac{\dim(E'_N(\varepsilon, l))}{1+\lambda}$$

we have

$$\Sigma \; |I_i|^{\mu} \geqq \Sigma \; |J_i|^{\mu(1+\lambda)} \geqq 1$$

and hence

$$\dim(\phi(E'_N(\varepsilon, l))) \geqq \frac{\dim(E'_N(\varepsilon, l))}{1+\lambda}.$$

As we let $l \to \infty$ we can take $\lambda \to 0$ so

$$\dim(\phi(\lim_{l\to\infty} E'_N(\varepsilon, l)) \geqq \dim\left(\lim_{l\to\infty} E'_N(\varepsilon, l)\right).$$

The first part of the proof is now completed by letting $\varepsilon \to 0$.

Next we will show that $\dim L(\gamma) \leqq \dim(E_{N+2})$ and this plus Jarnik's theorem (see [2]) that

$$\dim(E_N) \leqq 1 - \frac{1}{8N \log N}, \; N > 8$$

will complete the proof.

Clearly if $\xi = [x_1, x_2, \cdots]$ has infinitely many $x_i \geqq N+3$ then $M(\xi) > N+3$ so $\xi \notin L(\gamma)$. Thus $L(\gamma) \subset \bigcup_{\bar{k}\in K} G_{\bar{k}}$ where K is the set of all finite vectors $\bar{k} = (k_i, k_2, \cdots, k_{2n})$, each k_i being a positive integer, and $G_{\bar{k}}$ is the set of all $\xi = [x_1, x_2, \cdots]$ with

$$x_i = k_i, \qquad 1 \leqq i \leqq 2n,$$

$$x_i \leqq N+2, \qquad 2n < i.$$

Since K is countable it will be sufficient to show that $\dim(G_{\bar{k}}) \leqq \dim(E_{N+2})$ for each $\bar{k}$. We define an order preserving map Ψ of E_{N+2} onto $G_{\bar{k}}$ by setting, for $\xi = [x_1, x_2, \cdots]$ in E_{N+2}, $\Psi(\xi) = [k_i, k_2, \cdots, k_{2n}, x_1, x_2, \cdots]$. If also $\xi' = [x'_1, x'_2, \cdots]$ is in E_{N+2} and $x_i = x'_i$ for $i < l$ but $x_l \neq x'_l$ then, letting $N' = \max(k_1, \cdots, k_{2n}, N+2)$,

$$|\xi - \xi'| \geqq \frac{1 - (l_{N+2} - s_{N+2})}{4Q_l(\xi)Q_l(\xi')} \geqq A \frac{N'}{Q_l(\xi)Q_l(\xi')}$$

$$\geqq A \frac{N'}{Q_{2n+l}(\Psi(\xi))Q_{2n+l}(\Psi(\xi'))}$$

$$\geqq A \left| \Psi(\xi) - \Psi(\xi') \right|.$$

We have used here Lemma 4, the fact that $Q_l(\eta) \leqq Q_{2n+l}(\Psi(\eta))$, and (1) in that order. If (I_i) is a covering of E_{N+2} of norm δ, each I_i being the smallest interval containing $I_i \cap E_{N+2}$, then $(\Psi(I_i))$ is a covering of $G_{\bar{k}}$ of norm at most δ/A. For any λ less than $\dim(G_{\bar{k}})$ and for small enough δ we have

$$\Sigma|I_i|^\lambda \geqq A^\lambda \Sigma|\Psi(I_i)|^\lambda > 1$$

so $\dim(E_{N+2}) \geqq \dim(G_{\bar{k}})$.

Bibliography

1. M. Hall, *On the sum and product of continued fractions*, Ann. of Math. **48** (1947), 966–993.

2. V. Jarnik, *Zur Metrischen Theorie der Diophantischen Approximationen*, Prace Mat. Fix. **36** (1928), 91–106.

3. A. Ja. Khintchine, *Continued fractions*, Noordhoff, Groningen, The Netherlands, 1963.

4. J. Kinney and T. Pitcher, *The dimension of some sets defined in terms of F-expansions*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **4** (1966), 293–315.

5. O. Perron, *Über die Approximation Irrationaler Zahlen durch Rationale*, S.-B. Heidelberger Akad. Wiss. Math. Nat. Kl. (1921), 3–17.

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